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界面裂纹的 Cell-Based 光滑有限元法研究^{*}

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摘要:为提高求解含界面裂纹结构断裂参数的精度,基于界面断裂力学和 Cell-Based 光滑有限元法,提出了求解双材料界面裂纹断裂参数的 Cell-Based 光滑有限元法,给出了求解应力强度因子的光滑子域交互积分法,对含中心界面裂纹双材料无限板进行了模拟,并与 FEM 计算结果和解析解进行了对比,讨论了光滑子元数和单元个数与正则应力强度因子的关系及其收敛性.数值算例结果表明该方法具有很好的收敛性和精度,可为研究人员和工程师设计制造多层材料提供必要参考.

关键词:光滑有限元法;界面裂纹;应力强度因子;交互积分

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Research on Cell-Based Smoothed Finite Element Method of Interface Cracks

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Abstract: To improve the accuracy of the fracture parameters of interface fracture structure, based on the interfacial fracture mechanics and smoothed finite element method, Cell-Based smoothed finite element method was proposed to solve the fracture parameters of bi-material interface crack, and the smoothing cells of the interaction integral method was given to obtain the stress intensity factor. The central interface cracks of bi-material infinite plate were simulated, and the FEM calculation results and the analytical solution were compared. The relationships among the number of elements, the number of subdomain and normalization stress intensity factors as well as the convergence of the proposed method were discussed. Numerical example results show that this method has good convergence and high accuracy and can be designed and manufactured as necessary references for researchers and engineers in multilayer materials.

Key words: smoothed finite element method; interfacial cracks; stress intensity factors; interaction integral

随着科学技术的发展,航空航天、机械工程和生物医学等领域对多层材料(如复合材料层合板、粘接

接头、薄膜/基体系统)的需求日益增多.多层材料的整体力学特性和响应完全依赖于界面的性能.裂纹

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或类似裂纹缺陷往往出现在界面处,裂纹尖端的应力集中导致裂纹扩展或胶粘层脱黏.借助计算机模拟双材料界面裂纹能量释放率或应力强度因子^[1],可进一步得到界面裂纹力学性能失配及裂纹扩张机理,为研究人员和工程师预测材料的寿命及提升多层材料的应用空间奠定基础.

England^[2]和Rice^[3]的研究奠定了界面断裂力学的理论基础.对于含界面裂纹复杂结构的断裂参数的求解不得不借助于数值计算方法.Bjerkén^[4]采用FEM对双材料界面裂纹问题进行了研究.Belytschko^[5]等提出了研究界面裂纹问题的无网格法.Sukumár^[6]等和江守燕^[7]等基于扩展有限元,通过相互作用积分^[8]求解了双材料界面裂纹的应力强度因子.姚振汉等^[9]采用边界元对界面裂纹进行了模拟.Zhao^[10]和Gao^[11]等分别采用数值流形方法和无网格流形方法对双材料界面裂纹的断裂参数进行了解. Pathak 等^[12]基于无网格法和扩展有限元法对界面裂纹的应力强度因子进行模拟.可见,采用数值计算方法求解界面裂纹的断裂参数是目前解决界面裂纹问题的主要手段.基于位移有限元求解的位移解偏小;边界元的应用范围以存在相应微分算子的基本解为前提;无网格计算效率低;扩展有限元在包含不连续界面的单元中需对间断函数进行数值积分,采用高斯积分求解时会存在较大误差.为提高求解精度,Liu 等^[13]将无网格法中的光滑应变措施^[14]引入有限元,提出了光滑有限元法.光滑有限元法具有网格要求低、形函数简单、计算精度高等优点,目前已应用于很多领域^[15-16],但关于界面裂纹问题的光滑有限元法研究还未见报道.

本文基于光滑有限元法,结合界面断裂力学提出了求解双材料界面裂纹断裂参数的 Cell-Based 光滑有限元法,计算应力强度因子时采用互交积分法,对无限大含中心裂纹的双材料板进行了模拟,并与FEM求解结果和解析解进行了对比.

1 界面断裂力学

如图1所示,各向同性双材料界面裂纹问题, E_s 和 ν_s ($s=1,2$)分别为材料1和材料2的弹性模量和泊松比,受远场载荷 σ_{yy}^∞ 和 τ_{xy}^∞ 作用.基于线弹性界面断裂力学理论,复应力强度因子 $K^{[4]}$ 为:

$$K = K_I + iK_{II} \quad (1)$$

式中: K_I 和 K_{II} 分别为I型和II型裂纹应力强度因子, $i = \sqrt{-1}$.

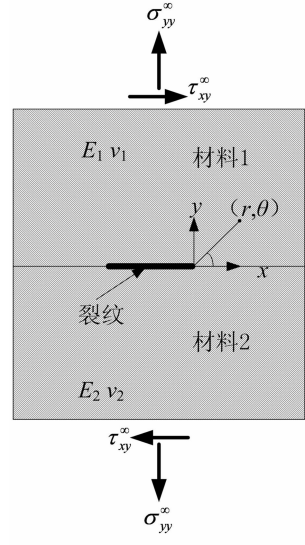


图1 双材料界面裂纹

Fig.1 Bi-material interface crack

裂纹尖端处的应力 σ_{yy} 和 τ_{xy} 有:

$$\sigma_{yy} + i\tau_{xy} = \frac{K}{\sqrt{2\pi r}} r^{-it} \quad (2)$$

式中: t 为双材料参数,表达式为

$$t = \frac{1}{2\pi} \log \left(\frac{1-\beta}{1+\beta} \right) \quad (3)$$

式中:

$$\beta = \frac{\mu_1(\kappa_2 - 1) - \mu_2(\kappa_1 - 1)}{\mu_1(\kappa_2 + 1) + \mu_2(\kappa_1 + 1)} \quad (4)$$

其中:

$$\kappa_s = \begin{cases} \frac{3-\nu_s}{1+\nu_s} & (\text{平面应力}) \\ 3-4\nu_s & (\text{平面应变}) \end{cases} \quad (5)$$

其中: μ_s 为材料 s 的剪切模量; κ_s 为材料 s 的Kolosov常数.

J 积分与应力强度因子的关系:

$$J = \frac{1}{E^*} \frac{|K|^2}{\cosh^2(\pi t)} = \frac{1}{E^*} \frac{K_I^2 + K_{II}^2}{\cosh^2(\pi t)} \quad (6)$$

式中:

$$\frac{2}{E^*} = \frac{1}{E_1} + \frac{1}{E_2} \quad (7)$$

$$\bar{E}_s = \begin{cases} E_s & (\text{平面应力}) \\ \frac{E_s}{1-\nu_s^2} & (\text{平面应变}) \end{cases} \quad (8)$$

双材料界面裂纹尖端上半面位移渐近解^[6]:

$$u_j = \frac{1}{2\mu_1} \sqrt{\frac{r}{2\pi}} \{ \text{Re} [K r^{it}] \bar{u}_j^I(\theta, t, \nu_1) + \text{Im} [K r^{it}] \bar{u}_j^{II}(\theta, t, \nu_1) \} \quad (j = x, y) \quad (9)$$

$$\begin{aligned} \bar{u}_x^I &= Q[-e^{2t\langle\pi-\theta\rangle} \left(\cos \frac{\theta}{2} + 2t \sin \frac{\theta}{2} \right) + \\ &\quad \kappa_1 \left(\cos \frac{\theta}{2} - 2t \sin \frac{\theta}{2} \right) + (1+4t^2) \sin \frac{\theta}{2} \sin \theta] \end{aligned} \quad (10)$$

$$\begin{aligned} \bar{u}_x^{II} &= Q[e^{2t\langle\pi-\theta\rangle} \left(\sin \frac{\theta}{2} - 2t \cos \frac{\theta}{2} \right) + \\ &\quad \kappa_1 \left(\sin \frac{\theta}{2} + 2t \cos \frac{\theta}{2} \right) + (1+4t^2) \cos \frac{\theta}{2} \sin \theta] \end{aligned} \quad (11)$$

$$\begin{aligned} \bar{u}_y^I &= Q[e^{2t\langle\pi-\theta\rangle} \left(\sin \frac{\theta}{2} - 2t \cos \frac{\theta}{2} \right) + \kappa_1 \left(\sin \frac{\theta}{2} + \right. \\ &\quad \left. 2t \cos \frac{\theta}{2} \right) - (1+4t^2) \cos \frac{\theta}{2} \sin \theta] \end{aligned} \quad (12)$$

$$\begin{aligned} \bar{u}_y^{II} &= Q[e^{2t\langle\pi-\theta\rangle} \left(\cos \frac{\theta}{2} + 2t \sin \frac{\theta}{2} \right) - \\ &\quad \kappa_1 \left(\cos \frac{\theta}{2} - 2t \sin \frac{\theta}{2} \right) + (1+4t^2) \sin \frac{\theta}{2} \sin \theta] \end{aligned} \quad (13)$$

式中: (r, θ) 为极点在右裂尖处的极坐标, Re 和 Im 分别为取复数的实部和虚部, $r^{it} = e^{it \log r} = \cos(t \log r) + i \sin(t \log r)$, Q 的表达式为:

$$Q = \frac{e^{-t\langle\pi-\theta\rangle}}{(1+4t^2) \cosh(\pi t)} \quad (14)$$

下半面位移渐近解将 $-\pi$ 代替 π , 用 κ_2 代替 κ_1 .

2 Cell-Based 光滑有限元法

该问题的光滑 Galerkin 弱形式可表示为:

$$\begin{aligned} \int_{\Omega} \delta \boldsymbol{\varepsilon}^T(\mathbf{u}) \mathbf{D} \boldsymbol{\varepsilon}(\mathbf{u}) d\Omega - \int_{\Omega} \delta \mathbf{u}^T \mathbf{b} d\Omega - \\ \int_{\Gamma} \delta \mathbf{u}^T \mathbf{f} d\Gamma = 0 \end{aligned} \quad (15)$$

式中: Ω 为求解域, δ 为变分符号, T 为矩阵的转置, $\boldsymbol{\varepsilon}$ 为光滑应变矩阵, \mathbf{D} 为弹性矩阵(与柔度矩阵互逆), \mathbf{u} 为广义位移, \mathbf{b} 为体力, \mathbf{f} 为力边界 Γ 上的面力.

将求解域 Ω 离散为 n_e 个四边形单元, n_d 个节点, $\Omega = \cup_{i=1}^{n_e} \Omega_i^e$, $\Omega_i^e \cap \Omega_j^e = \emptyset, i \neq j, \emptyset$ 为空集, 再将 Ω_i^e 划分为 n_c 个光滑子域, 如图 2 所示, $n_c=4$ 时光滑子域的离散, \bullet 为节点, \square 为光滑节点, \circ 为高斯点, $(N_1 \ N_2 \ N_3 \ N_4)$ 为该点处的形函数值.

广义位移为:

$$\mathbf{u} = \sum_{i=1}^{n_e} \mathbf{N}_i \mathbf{q}_i \quad (16)$$

式中: \mathbf{q}_i 为节点位移, \mathbf{N}_i 为形函数对角矩阵.

光滑应变为:

$$\bar{\boldsymbol{\varepsilon}}(\mathbf{x}_c) = \int_{\Omega_c} \boldsymbol{\varepsilon}(\mathbf{x}) \boldsymbol{\Phi}(\mathbf{x} - \mathbf{x}_c) d\Omega \quad (17)$$

式中: $\boldsymbol{\varepsilon}(\mathbf{x})$ 为应变矩阵, $\boldsymbol{\Phi}(\mathbf{x} - \mathbf{x}_c)$ 为光滑函数, 取

$$\boldsymbol{\Phi}(\mathbf{x} - \mathbf{x}_c) = \begin{cases} 1/A_c & \mathbf{x} \in \Omega_c \\ 0 & \mathbf{x} \notin \Omega_c \end{cases} \quad (18)$$

式中: A_c 为第 c 光滑区域的面积, $A_c = \int_{\Omega_c} d\Omega$.

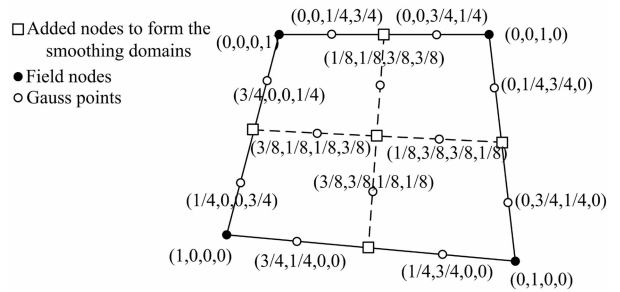


图 2 光滑域的划分

Fig. 2 Division of an element into smoothing cells

将式(18)代入式(17), 由散度定理得

$$\bar{\boldsymbol{\varepsilon}}(\mathbf{x}_c) = \frac{1}{2A_c} \int_{\Gamma_c} \mathbf{u} \mathbf{m} d\Gamma \quad (19)$$

式中: Γ_c 为光滑域 Ω_c 的边界, \mathbf{m} 为积分段外法向向量.

将式(16)代入式(19)中, 可得

$$\bar{\boldsymbol{\varepsilon}}(\mathbf{x}_c) = \sum_{i=1}^{n_e} \mathbf{B}_i(\mathbf{x}_c) \mathbf{q}_i \quad (20)$$

式中:

$$\mathbf{B}_i(\mathbf{x}_c) = \frac{1}{A_c} \sum_{b=1}^{n_b} \begin{pmatrix} N_i(\mathbf{x}_b^G) m_x & 0 \\ 0 & N_i(\mathbf{x}_b^G) m_y \\ N_i(\mathbf{x}_b^G) m_y & N_i(\mathbf{x}_b^G) m_x \end{pmatrix} l_b^i \quad (21)$$

式中: \mathbf{x}_b^G 和 l_b^i 分别为光滑边界 Γ_b 的中点(高斯点)和长度, n_b 为每个光滑单元的边界总数, m_x 和 m_y 分别为外法向向量 \mathbf{m} 沿 x 和 y 方向的分量.

式(20)和式(16)代入式(15)中, 可得离散方程

$$\mathbf{K} \mathbf{q} = \mathbf{F} \quad (22)$$

式中: \mathbf{K} 为整体光滑刚度矩阵, 可由光滑单元刚度矩阵组装得到

$$\mathbf{K}_{ij} = \sum_{k=1}^{n_s} \mathbf{B}_i^T \mathbf{D} \mathbf{B}_j A_k \quad (23)$$

式中: n_s 为光滑子域的总数, \mathbf{F} 为力向量,

$$\mathbf{F} = \int_{\Omega} \boldsymbol{\varphi}^T \mathbf{b} d\Omega - \int_{\Gamma} \boldsymbol{\varphi}^T \mathbf{f} d\Gamma \quad (24)$$

式中: $\boldsymbol{\varphi}$ 为有限元形函数.

Cell-Based 光滑有限元法推导过程中未对弹性矩阵 \mathbf{D} 做任何约束,因此,Cell-Based 光滑有限元法不仅适用于各向同性材料,同样适用于各向异性材料.

3 光滑子域交互积分 M

如图 3 所示,交互积分中,考虑两种独立的平衡状态,状态 1 ($\sigma_{ij}^{(1)}, \epsilon_{ij}^{(1)}, u_i^{(1)}$) 为真实场状态,状态 2 ($\sigma_{ij}^{(2)}, \epsilon_{ij}^{(2)}, u_i^{(2)}$) 为辅助场状态,其表达式^[5]为:

$$M^{(1,2)} = - \int_S \left[-\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} - \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} + \sigma_{ik}^{(1)} \epsilon_{ik}^{(2)} \delta_{1j} \right] \frac{\partial g}{\partial x_j} dS$$

$$i = x, y, j = x, y, k = x, y \quad (25)$$

式中: δ_{ij} 为克罗内克函数; S 为由曲线 Γ, C^+, C^- 和 C^0 围成积分区域;在 Γ 上单位外法向量 $T_j = -m_j$,在 C^+, C^- 和 C^0 上 $T_j = m_j$; g 为权函数.

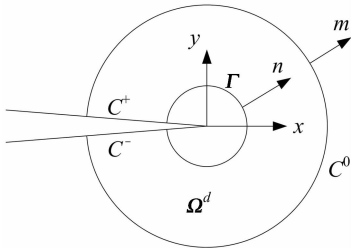


图 3 交互积分 M
Fig. 3 Interaction integral M

采用 Cell-Based 光滑有限元法求解交互积分 $M^{(1,2)}$ 时,其表达式为:

$$M^{(1,2)} = \sum_{m=1}^{n_d} \int_{\Omega_m} \left[\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - \sigma_{ik}^{(1)} \epsilon_{ik}^{(2)} \delta_{1j} \right] \frac{\partial g}{\partial x_j} d\Omega$$

$$(26)$$

式中: n_d 为积分区域单元的个数,如图 4 所示. Ω_m 包含 $\Omega_1^m, \Omega_2^m, \Omega_3^m$ 和 Ω_4^m 等 4 个光滑子元,在交互积分的等效区域积分中,如果单元 e 的节点 x_i 未处于 Ω_m 内,则 $g=0$;如果节点 x_i 处于 Ω_m 内,则 $g=1$.

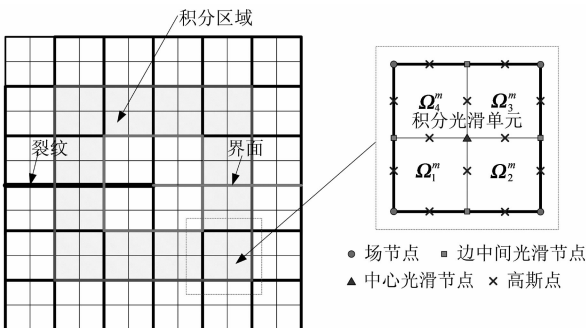


图 4 光滑子域交互积分 M
Fig. 4 Smoothing cells of interaction integral M

由式(6)可知交互积分 $M^{(1+2)}$ 与应力强度因子的关系^[6]为:

$$M^{(1,2)} = \frac{2}{E^* \cosh^2(\pi t)} [K_I K_I^{aux} + K_{II} K_{II}^{aux}] \quad (27)$$

式中: K_I^{aux} 和 K_{II}^{aux} 为辅助场下的 I 型和 II 型应力强度因子.

取 $K_I^{aux} = 1, K_{II}^{aux} = 0$, 这时, $M^{(1,2)} = M_1^{(1,2)}$, 式(27)得

$$K_I = \frac{E^* \cosh^2(\pi t)}{2} M_1^{(1,2)} \quad (28)$$

取 $K_I^{aux} = 0, K_{II}^{aux} = 1$, 这时, $M^{(1,2)} = M_2^{(1,2)}$, 式(27)得

$$K_{II} = \frac{E^* \cosh^2(\pi t)}{2} M_2^{(1,2)} \quad (29)$$

4 数值算例

取一含中心界面裂纹无限大板,如图 5 所示,材料参数 $E_1 = 10 \text{ GPa}, E_2 = 220 \text{ GPa}, \nu_1 = 0.3, \nu_2 = 0.2571, h = l = 30 \text{ mm}$, 裂纹长度为 $2a, \sigma^\infty = 1 \text{ MPa}$, 平面应变问题,应变能可定义如下:

$$\|u\|_{E(\Omega)} = \left(\frac{1}{2} \int_{\Omega} \boldsymbol{\epsilon}^T \mathbf{D} \boldsymbol{\epsilon} d\Omega \right)^{\frac{1}{2}} \quad (30)$$

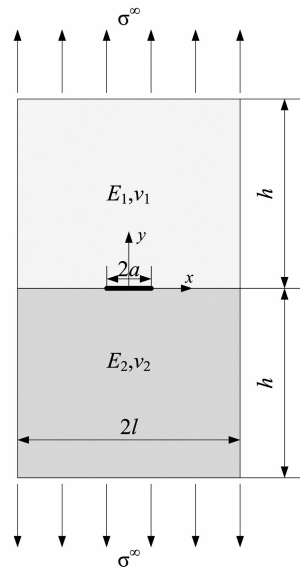


图 5 中心界面裂纹板受拉伸作用
Fig. 5 Centre interface cracks plate under tension

正则应力强度因子的解析解^[3]如下:

$$\frac{K_I}{K_0} = 1.008, \frac{K_{II}}{K_0} = 0.1097 \quad (31)$$

式中:

$$K_0 = \sigma_{yy}^\infty \sqrt{\pi a} \quad (32)$$

式中: σ_{yy}^∞ 为无穷远处沿 y 方向正应力载荷.

由于结构和载荷对称,取右侧 1/2 模型,边界条件为约束左右两边水平向位移以及右下角竖向位移,离散成 8 400 个单元,每个单元划分为 n_c 个光滑子元,表 1 给出了 $a=1$ mm 时,由 Cell-SFEM 计算得到的应变能和正则应力强度因子.由表 1 可见,本文方法具有很好的收敛性,光滑子元数为 4 时就可准确地求解界面裂纹右侧的正则应力强度因子.

表 1 光滑子元个数与应变能和正则应力强度因子的关系
Tab. 1 The relationship between number of smoothing cells and strain energy and normalization stress intensity factors

光滑子元 n_c	$\ u\ _{E(\Omega)}/J$	K_I/K_0	K_{II}/K_0
2	3.8013×10^{-5}	0.985 3	0.114 6
4	3.8995×10^{-5}	0.994 4	0.112 6
6	3.8993×10^{-5}	1.001 2	0.112 0
8	3.8990×10^{-5}	1.003 1	0.111 8
10	3.8991×10^{-5}	1.005 9	0.110 8

图 6 给出了 1/2 模型的应力 σ_{yy} 等势线和应变 ϵ_{yy} 等势线.可见,应力场几乎是连续的和关于 x 轴对称的,则应变场是不连续的,这种不连续是由于材料属性改变引起的.图 6 中应力场和应变场满足界面应力和应变分布情况.

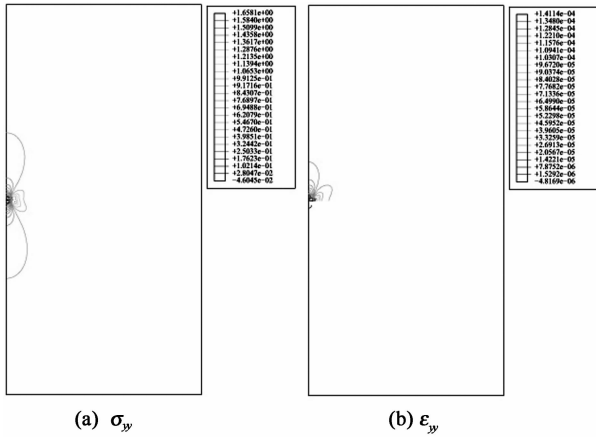


图 6 1/2 模型的应力 σ_{yy} 等势线和应变 ϵ_{yy} 等势线

Fig. 6 Stress σ_{yy} and strain ϵ_{yy} of the equipotential line in half model

图 7 和图 8 给出了裂纹长度为 $a=1$ mm 时,光滑子元个数为 4 时 Cell-SFEM 所求解的正则应力强度因子,并与 FEM 计算结果和解析解做了对比.图 9 和图 10 给出了裂纹长度 $a=0.4$ mm, 0.6 mm, 0.8 mm, 1.0 mm 和 1.2 mm 时,结构离散为 7 200 个单元,每个单元划分为 4 个光滑子元时,Cell-SFEM 所求解的正则应力强度因子,并与 FEM 计算结果和解析解做了对比.由此可见,在相同单元数下,Cell-SFEM 的计算精度高于 FEM.

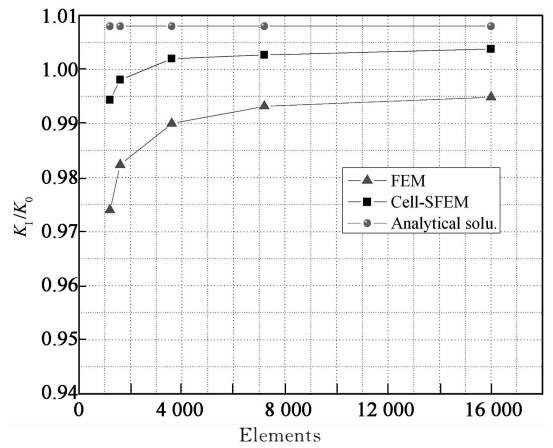


图 7 不同单元下由 FEM,Cell-SFEM 和解析解所得正则应力强度因子 K_I/K_0 结果比较

Fig. 7 Comparison of normalization stress intensity factor K_I/K_0 using FEM,Cell-SFEM and analytical solutions under different elements

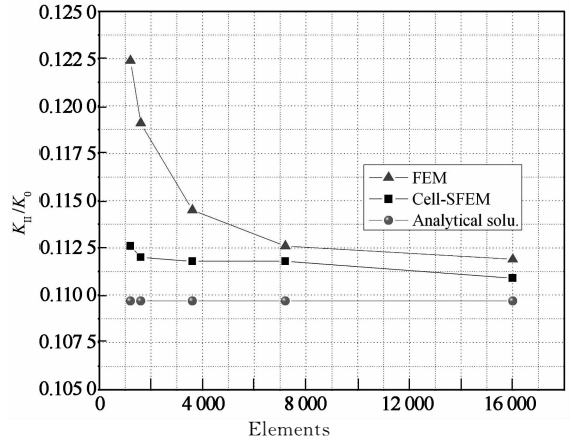


图 8 不同单元下由 FEM,Cell-SFEM 和解析解所得正则应力强度因子 K_{II}/K_0 结果比较

Fig. 8 Comparison of normalization stress intensity factor K_{II}/K_0 using FEM,Cell-SFEM and analytical solutions under different elements

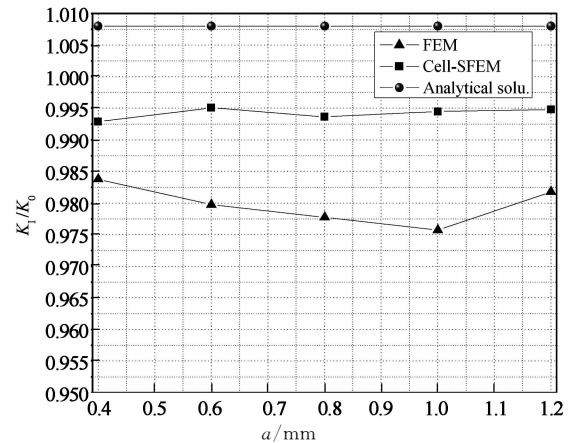


图 9 不同裂纹长度下由 FEM,Cell-SFEM 和解析解所得正则应力强度因子 K_I/K_0 结果比较

Fig. 9 Comparison of normalization stress intensity factor K_I/K_0 using FEM,Cell-SFEM and analytical solutions under different crack lengths

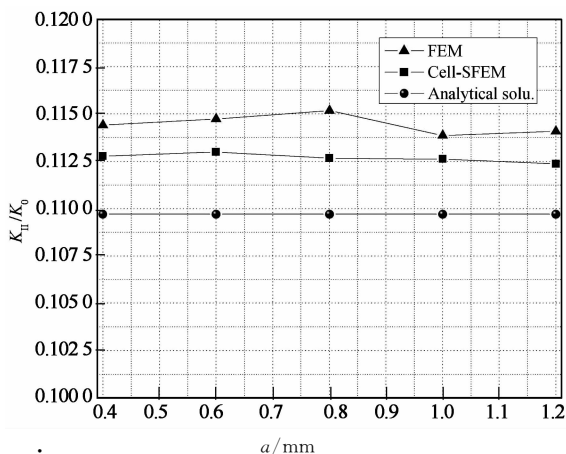


图 10 不同裂纹长度下由 FEM, Cell-SFEM 和解析解所得正则应力强度因子 K_{II}/K_0 结果比较

Fig. 10 Comparison of normalization stress intensity factor K_{II}/K_0 using FEM, Cell-SFEM and analytical solutions under different crack lengths

5 结论

本文提出求解含界面裂纹问题的 Cell-Based 光滑有限元法,对无限大含中心界面裂纹的双材料板进行了模拟,并与 FEM 计算结果和解析解进行了对比,得到以下结论:

- 1) 在相同单元数下, Cell-SFEM 的计算精度高于 FEM.
- 2) Cell-SFEM 具有很好的收敛性,光滑子元取 4 时就具有了较高的求解精度.
- 3) 基于 Cell-SFEM 的交互积分 M 求解简单.

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