

层状地基与弹性薄板相互作用的边界元解*

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摘要:将无限大薄板的基本解作为薄板边界积分方程的核函数,对薄板的内部和边界进行离散,并假定薄板内部和边界上的节点与地基反力的分布情况,得到薄板的边界元方程组;同时基于层状地基的解析层元解,通过 Gauss-Legendre 积分得到地基柔度矩阵;结合地基与薄板接触面上的位移协调条件,得到层状地基与薄板共同作用问题总的边界元法方程组;求解该方程组,得到层状地基与薄板共同作用问题的解答.基于本文理论,编制了相应的 FORTRAN 程序,通过与已有文献结果对比验证本文理论及程序的正确性,数值分析结果表明:方形基础薄板情况下,离板中心越近,垂直于坐标轴 $y(x)$ 方向、距离相等的 2 条线段的竖向位移差越小,且该位移差随着板-土刚度比减小而减小;随着板长宽比的增大,板中心点与长边中点位移差变化不明显,而短边中心与边界角点的位移差也有相类似的规律.

关键词:边界元;层状地基;薄板;解析层元

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A BEM for Interaction between Layered Foundations and an Elastic Thin Plate

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Abstract: The kernel functions of the boundary integral equations for thin plate were determined by the fundamental solutions for an infinite thin plate. By the discretization of the plate interior and boundary as well as the assumption of the distribution states of plate nodes and foundation reaction forces, the BEM equations of the plate can be established. Meanwhile, based on the analytical layer element solutions for layered foundations, the flexibility matrix of the foundation was obtained by a two-dimensional Gauss-Legendre quadrature. Taking into account the compatible conditions of the displacements at the soils-plate interface, the global BEM equations for the interaction problem between the layered foundation and the thin plate were then established. The solutions for the problem were further obtained by solving the global BEM equations. The accuracy of the present method was verified by comparing existing solutions with the numerical results obtained from the corresponding FORTRAN program in this study. It is observed from numerical examples that when a square thin plate is placed on a foundation, the settlement difference between the two lines perpendicular to y or x coordinate decreases as they approach the center of the plate.

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and the difference decreases with the decrease of the plate-soil stiffness ratio. Furthermore, the settlement discrepancy between the plate center and the midpoint of the long side is unapparent with the increasing length-width ratio, and the similar variation trend can be found between the midpoint of the wide side and angular point.

Key words: boundary element; layered soils; thin plates; analytical layer element

筏板基础具有刚度大、整体性好、能较好地抵抗不均匀沉降的优点,因此在高层建筑中得到了广泛的应用。目前,地基与板相互作用分析的方法主要有:有限差分法^[1]、有限单元法^[2-4]、边界单元法^[5]、边界单元-有限单元耦合法^[6]、广义微分求积法^[7]、半解析数值方法^[8-10],以及有限网格法^[11]等。相比于有限元、有限层等方法,边界单元法能将求解过程的维数降低一维,并具有计算时间短、精度高等优点。因此,很多学者运用边界元法来研究筏板与地基的相互作用问题。余颖禾和朱万宁^[12]将地基效应归并到地基板的弯曲微分方程内,得到了含有第三类复变量的 Bessel 函数的基本解,再根据该问题的边界积分方程,建立了 Winkler 和双参数地基上薄板的无奇异边界单元法。王建国和黄茂光^[13]提出了双参数地基上薄板问题的边界单元解法。邓安福等^[14]采用边界单元法研究了双参数地基上的厚板问题。Rashed 等^[15]通过边界元法研究了 Winkler 地基上的厚板问题。闫富有等^[16]基于 Reissner 板的边界积分方程,建立了有限压缩层地基上厚筏基础与地基相互作用分析的边界元法。

由以上研究可知,目前板-土相互作用的边界单元法研究所采用的地基模型大多是 Winkler 和双参数地基模型。Winkler 地基模型将地基对板的作用看做是一系列相互独立的弹簧,忽略了弹簧之间的剪切作用,因此只适用于很软弱的地基土。双参数地基模型虽然在独立弹簧之间引入力学的相互作用以消除其不连续性,但其参数较难获取,因而限制了它的工程应用。天然地基由于沉积而常常呈层状分布,因此采用层状地基模型更加符合工程实际。而层状地基上板的边界元研究还很少见诸报道。为此本文对层状地基与弹性薄板的共同作用问题进行边界元分析,以便精确、高效地求解基础板问题。

1 弹性薄板的边界积分方程

考虑横向分布荷载 p 时,弹性薄板的控制方程为:

$$D \nabla^2 \nabla^2 s = p. \quad (1)$$

式中: s 为板中面的横向位移; ∇^2 为 Laplace 算子; $D = E_p h_p^3 / [12(1 - \nu_p^2)]$, 为薄板的抗弯刚度,其中, h_p , E_p 和 ν_p 分别为板的厚度、弹性模量和泊松比。

由式(1)可得无限大平面薄板(如图 1 所示)的基本解为^[17]:

$$s^*(\xi_2, \xi_1) = \frac{1}{8\pi} r^2 \ln r, \quad (2)$$

$$V_n^*(\xi_2, \xi_1) = -\frac{D}{4\pi r} \cos \beta \{2 + (1 - \nu_p) \cos 2\beta\} + \frac{D(1 - \nu_p)}{4\pi r} \cos 2\beta, \quad (3)$$

$$M_n^*(\xi_2, \xi_1) = -\frac{D}{8\pi} \{(1 - \nu_p) \cos 2\beta + 2(1 + \nu_p)(\ln r + 1)\}, \quad (4)$$

$$\theta_n^*(\xi_2, \xi_1) = \frac{1}{8\pi} (2 \ln r + 1) r \cos \beta. \quad (5)$$

式中: $s^*(\xi_2, \xi_1)$, $V_n^*(\xi_2, \xi_1)$, $M_n^*(\xi_2, \xi_1)$ 和 $\theta_n^*(\xi_2, \xi_1)$ 对应于在 $\xi_1(x_1, y_1)$ 处无限大平板作用大小为 D 的横向集中荷载时在 $\xi_2(x_2, y_2)$ 处产生的横向位移、剪力、弯矩和转角, $\xi_1(x_1, y_1)$ 和 $\xi_2(x_2, y_2)$ 分别为基本解的源点和场点; $r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, 为 ξ_1 与 ξ_2 两点之间的距离。

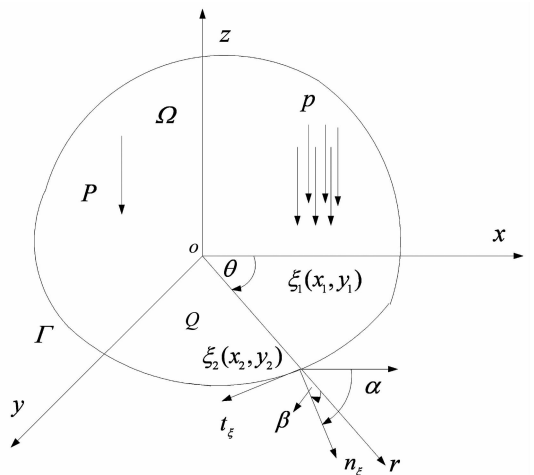


图 1 薄板边界元法示意图

Fig. 1 The diagram of the BEM for a thin plate

采用由式(2)~(5)得到的基本解作为边界积分方程的核函数,于是可以得到具有光滑边界的薄板

挠度边界积分方程:

$$D_p \cdot c(\xi_1) \cdot s(\xi_1) = \int_{\Omega} p(\xi_2) \cdot s^*(\xi_2, \xi_1) d\Omega + \int_{\Gamma} [s^*(\xi_2, \xi_1) \cdot V_n^*(\xi_2) - s(\xi_2) \cdot V_n^*(\xi_2, \xi_1)] d\Gamma + \int_{\Gamma} [\theta(\xi_2) \cdot M_n^*(\xi_2, \xi_1) - M_n(\xi_2) \cdot \theta_n^*(\xi_2, \xi_1)] d\Gamma. \quad (6)$$

式中:当源点在板内部时,即 $\xi_1 \in \Omega$ 时, $c(\xi_1) = 1$, 当源点在板边界时,即 $\xi_1 \in \Gamma$ 时, $c(\xi_1) = 0.5$; M_n , V_n 和 θ 分别表示薄板边界的弯矩、横向力、转角; Γ 和 Ω 分别表示薄板的边界域和内域; α 为 n_ξ 与 x 的夹角, θ 为幅角, $\beta = \alpha - \theta$, 其中, t_ξ 和 n_ξ 分别为 ξ_2 点处的切线方向和外法线方向.

2 层状地基与薄板的共同作用

如图 2 和图 3 所示, L_p 和 b_p 分别为薄板的长度和宽度; z_i , ν_i 和 E_i ($i = 1, 2, \dots, n$) 分别为第 i 层土的下表面深度、泊松比和弹性模量; 板边界划分为 N_e 个单元, 边界节点 $\xi_i^c(x_i^c, y_i^c)$ ($i = 1, 2, \dots, N_e$) 取为边界单元的中心, 假设各边界单元具有相同的转角和竖向位移; 薄板内域划分成 N_c 个单元, 并假设每个单元内地基反力 q_i ($i = 1, 2, \dots, N_c$) 均匀分布, 每个板单元内具有相同的竖向位移, 计算节点 $\xi_i^c(x_i^c, y_i^c)$ ($i = 1, 2, \dots, N_c$) 取为单元中心.

假设板四边自由, 有:

$$V_n(\xi_2) = 0, \quad (7)$$

$$M_n(\xi_2) = 0. \quad (8)$$

结合地基反力分布假设和式(7)与式(8), 当源点为内点及边界点时, 式(6)可分别表示为式(9)和式(10), 即:

$$D \cdot s^c(\xi_2) - \sum_{i=1}^{N_e} \int_{\Gamma_i} \theta(\xi_2) \cdot M_n^*(\xi_2, \xi_1) d\Gamma_i + \sum_{i=1}^{N_e} \int_{\Gamma_i} s(\xi_2) \cdot V_n^*(\xi_2, \xi_1) d\Gamma_i = \int_{\Omega} p(\xi_2) \cdot s^*(\xi_2, \xi_1) d\Omega - \sum_{i=1}^{N_c} \int_{\Omega_i} q_i(\xi_2) \cdot s^*(\xi_2, \xi_1) d\Omega_i, \quad (9)$$

$$\frac{D}{2} \cdot s^c(\xi_2) - \sum_{i=1}^{N_e} \int_{\Gamma_i} \theta(\xi_2) \cdot M_n^*(\xi_2, \xi_1) d\Gamma_i + \sum_{i=1}^{N_e} \int_{\Gamma_i} s(\xi_2) \cdot V_n^*(\xi_2, \xi_1) d\Gamma_i =$$

$$\int_{\Omega} p(\xi_2) \cdot s^*(\xi_2, \xi_1) d\Omega - \sum_{i=1}^{N_c} \int_{\Omega_i} q_i(\xi_2) \cdot s^*(\xi_2, \xi_1) d\Omega_i. \quad (10)$$

式中: Γ_i ($i = 1, 2, \dots, N_e$) 对应于各段边界的单元; Ω_i 对应于反力 q_i ($i = 1, 2, \dots, N_c$) 的作用面积; s^c 和 s^e 分别表示薄板内点、边界点的竖向位移.

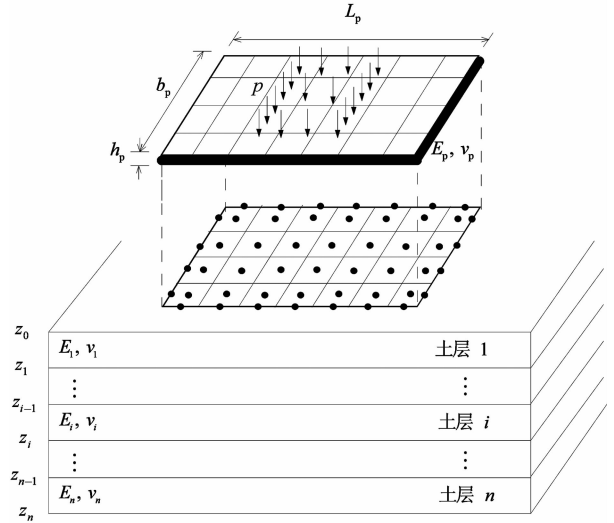


图 2 层状地基与薄板相互作用图
Fig. 2 The draft of the interaction between layered soils and a thin plate

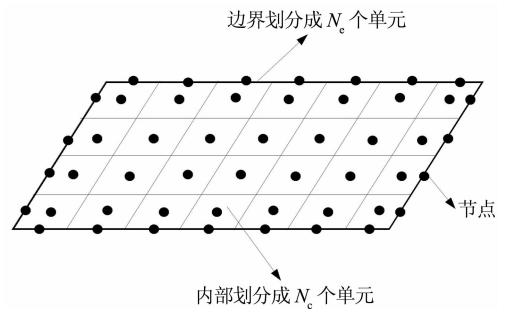


图 3 弹性薄板单元及节点图
Fig. 3 The elements and nodes of an elastic thin plate

当板上作用均布荷载 p 时, 薄板的边界积分方程可表示为:

$$\begin{bmatrix} \mathbf{A}_{N_e \times N_e}^1 & \mathbf{A}_{N_e \times N_e}^2 & \mathbf{B}_{N_e \times N_e}^1 & \mathbf{0}_{N_e \times N_e} \\ \mathbf{A}_{N_c \times N_e}^3 & \mathbf{A}_{N_c \times N_e}^4 & \mathbf{B}_{N_c \times N_e}^2 & \mathbf{I}_{N_c \times N_e} \end{bmatrix} \times \begin{bmatrix} \mathbf{s}_{N_e \times 1}^e \\ \boldsymbol{\theta}_{N_e \times 1} \\ \mathbf{q}_{N_c \times 1} \\ \mathbf{s}_{N_c \times 1}^c \end{bmatrix} = p \cdot \begin{bmatrix} \mathbf{C}_{N_e \times 1}^1 \\ \mathbf{C}_{N_e \times 1}^2 \end{bmatrix}. \quad (11)$$

其中,

$$\mathbf{A}_{ij}^1 = \int_{\Gamma_j} V_n^*(\xi_j^c, \xi_i^c) d\Gamma_j + \frac{D}{2} \delta_{ij}; \quad (12)$$

$$A_{ij}^2 = \int_{\Gamma_j} M_n^* (\xi_j^c, \xi_i^c) d\Gamma_j; \quad (13)$$

$$A_{ij}^3 = \int_{\Gamma_j} V_n^* (\xi_j^c, \xi_i^c) d\Gamma_j; \quad (14)$$

$$A_{ij}^4 = \int_{\Gamma_j} M_n^* (\xi_j^c, \xi_i^c) d\Gamma_j; \quad (15)$$

$$B_{ij}^1 = \int_{\Omega_j} \tau w^* (\xi_j^c, \xi_i^c) d\Omega_j; \quad (16)$$

$$B_{ij}^2 = \int_{\Omega_j} \tau w^* (\xi_j^c, \xi_i^c) d\Omega_j; \quad (17)$$

$$C_i^1 = \int_{\Omega_j} \tau w^* (\xi_j^c, \xi_i^c) d\Omega_j; \quad (18)$$

$$C_i^2 = \int_{\Omega_j} \tau w^* (\xi_j^c, \xi_i^c) d\Omega_j; \quad (19)$$

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j; \end{cases} \quad (20)$$

$$s^e = [s_1^e \quad s_2^e \quad \dots \quad s_{N_e}^e]^T; \quad (21)$$

$$\theta = [\theta_1 \quad \theta_2 \quad \dots \quad \theta_{N_e}]^T; \quad (22)$$

$$q = [q_1 \quad q_2 \quad \dots \quad q_{N_e}]^T; \quad (23)$$

$$s^c = [s_1^c \quad s_2^c \quad \dots \quad s_{N_e}^c]^T. \quad (24)$$

式中: s^e, θ, s^c, q 分别为薄板边界节点位移、边界节点转角、内节点位移、地基反力矩阵; I^c 为元素是 D 的 N_e 阶对角矩阵。

根据层状各向同性弹性地基的解析层元解^[18], 通过两维 Gauss-Legendre 积分可得单位均布荷载作用在反力 q_i 区域时, 板节点 i 处的沉降为:

$$\delta_{ij} = \iint \delta(i, \phi, R) d\Gamma_j. \quad (25)$$

式中: $\delta(i, \phi, R)$ 为地基的基本解, 它表示层状地基表面任意点 ϕ 作用单位点荷载在点 i 引起的沉降, R 为点 ϕ 到点 i 的距离。

地基反力引起的板内点及边界点的竖向位移可表示为:

$$w^e = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N_e} \end{bmatrix} = \zeta^e \cdot q = \begin{bmatrix} \zeta_{11}^e & \zeta_{12}^e & \dots & \zeta_{1N_e}^e \\ \zeta_{21}^e & \zeta_{22}^e & \dots & \zeta_{2N_e}^e \\ \vdots & \vdots & & \vdots \\ \zeta_{N_e 1}^e & \zeta_{N_e 2}^e & \dots & \zeta_{N_e N_e}^e \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{N_e} \end{bmatrix}, \quad (26)$$

$$w^e = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N_e} \end{bmatrix} = \zeta^c \cdot q = \begin{bmatrix} \zeta_{11}^c & \zeta_{12}^c & \dots & \zeta_{1N_e}^c \\ \zeta_{21}^c & \zeta_{22}^c & \dots & \zeta_{2N_e}^c \\ \vdots & \vdots & & \vdots \\ \zeta_{N_e 1}^c & \zeta_{N_e 2}^c & \dots & \zeta_{N_e N_e}^c \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{N_e} \end{bmatrix}. \quad (27)$$

假设加载前后板与地基不脱离, 则有:

$$w^c = s^c, \quad (28)$$

$$w^c = s^c. \quad (29)$$

将式(26)–(29)代入式(11), 可得:

$$\begin{bmatrix} A_{N_e \times N_e}^2 & A_{N_e \times N_e}^1 \zeta^c + B_{N_e \times N_e}^1 \\ A_{N_e \times N_e}^4 & A_{N_e \times N_e}^3 \zeta^c + B_{N_e \times N_e}^2 + D \zeta^c \end{bmatrix} \begin{bmatrix} \theta_{N_e \times 1} \\ q_{N_e \times 1} \end{bmatrix} = p \begin{bmatrix} C_{N_e \times 1}^1 \\ C_{N_e \times 1}^2 \end{bmatrix}. \quad (30)$$

求解式(30), 得到地基与弹性薄板相互作用的板边界节点转角和地基反力, 再通过式(26)和(27)可求得薄板内部点和边界点的竖向位移。

3 数值计算与分析

3.1 理论与程序验证

为了验证本文理论及程序的正确性, 将本文的计算结果与 Wang & Cheung^[4]的结果进行对比(如图 4 所示), 其中, $b_p = 4.0 \text{ m}$, $\nu_s = 0.4$, $h_p = 0.2 \text{ m}$, $E_s = 0.343 \times 10^3 \text{ MPa}$, $\nu_p = 0.167$, $E_p = 0.343 \times 10^5 \text{ MPa}$, 均布荷载 $p = 0.98 \text{ MPa}$ (见图 5). 本文采用厚度为 $1\ 000 \text{ m}$ 的土层来模拟弹性半空间地基. 由图 5 可知, 本文的结果与 Wang & Cheung^[4]的结果吻合较好, 这表明本文理论与程序的正确性。

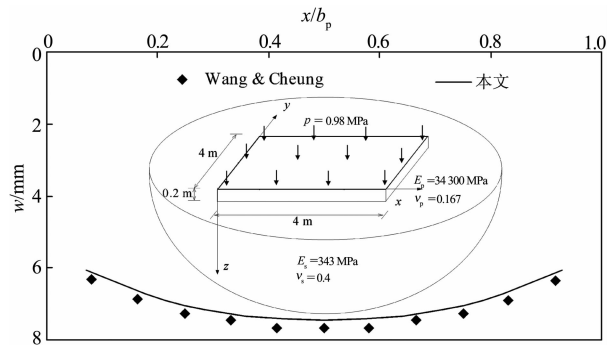


图 4 方形基础薄板边界的挠度对比图

Fig. 4 The comparison of vertical displacements of the boundary of a square foundation thin plate

3.2 板-土刚度比的影响

本例对弹性半空间上方形薄板在均布荷载作用下的挠度进行研究(如图 4 所示). 为分析方便, 将板-土刚度比定义为: $K = 4E_p(1 - \nu_s^2)h_p^3/[3E_s(1 - \nu_p^2)b_p^3]$; 而无量纲竖向位移取为: $I_1 = E_s w/[pb_p(1 - \nu_s^2)]$. 本例取土层厚度为 $250b_p$ 来模拟半空间, 对比分析 $K=0.05$ 和 $K=0.01$ 两种情况下板的竖向位移响应. 为方便描述, 板上各点以及线段的定义如图 6 所示, 薄板上不同线段的挠度曲线如图 7 所示. 由图 7 可知, 越接近板中心, 垂直于 y 方向的相邻 2 条线段的挠度差越小; 且其挠度差随 K 值减小而

减小。

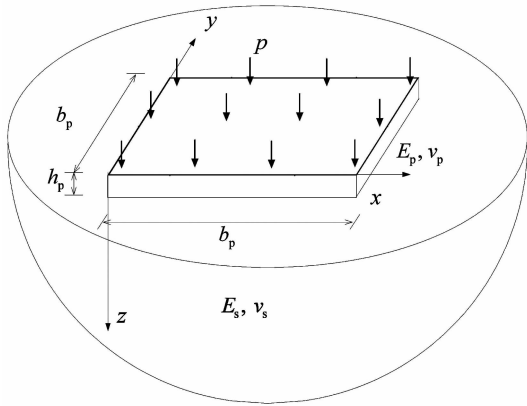


图5 弹性半空间上的方形薄板

Fig. 5 A square thin plate on an elastic half-space

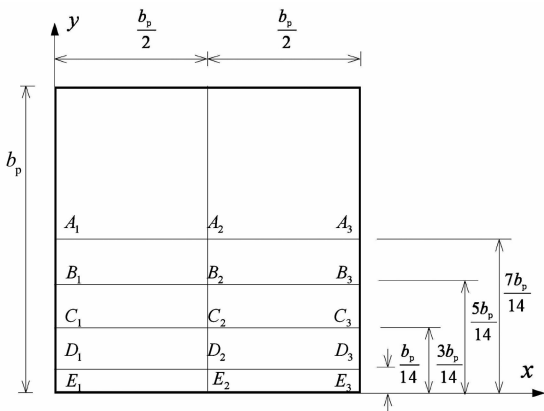


图6 方形薄板平面图

Fig. 6 The plane graph of a square thin plate

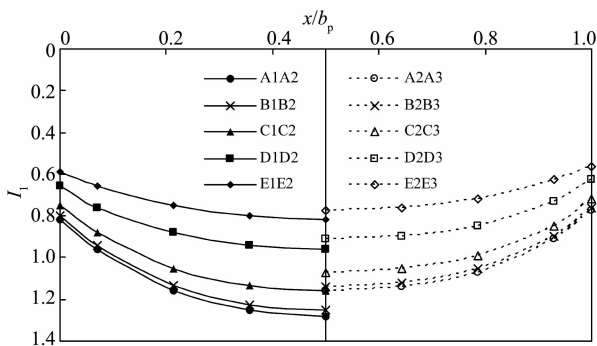


图7 当 \$K=0.05\$ (实线) 和 \$0.01\$ (虚线) 时各线段的竖向位移曲线

Fig. 7 The settlement curves of the lines on the plate when \$K=0.05\$ (solid line) and \$0.01\$ (imaginary line)

3.3 矩形板长宽比 \$\eta\$ 的影响

本例研究 3 层地基上矩形薄板在均布荷载 \$p\$ 作用下的沉降响应(如图 8 所示),其中, \$E_p : E_1 : E_2 : E_3 = 1\ 000 : 2.5 : 1 : 2\$, \$\nu_p = 0.2\$, \$\nu_1 = \nu_2 = \nu_3 = 0.35\$, \$h_1 : h_2 : h_3 = 1 : 1 : 2\$, \$h_i (i = 1, 2, 3)\$ 为第

\$i\$ 层土的厚度,其他参数的意义同图 2. 为分析矩形板长宽比 \$\eta\$ 的影响,本例对比分析以下 4 种工况:工况 1: \$\eta = 1\$;工况 2: \$\eta = 1.25\$;工况 3: \$\eta = 1.5\$;工况 4: \$\eta = 2\$. 计算结果如图 9 所示,其中, \$I_w = E_1 \omega / [p b_p (1 - \nu_p^2)]\$ 为无量纲竖向位移. 由图 9 可见,随着长宽比 \$\eta\$ 的增大,板各点竖向位移增大,短边中点与长边中点的挠度差也增大;但板中心点与长边中点位移差值随 \$\eta\$ 变化不明显,而且短边中心与边界角点位移差也有相类似的规律.

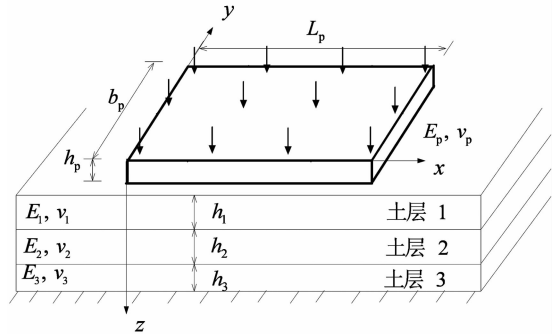


图8 三层地基上矩形薄板

Fig. 8 A rectangular plate on three-layered soils

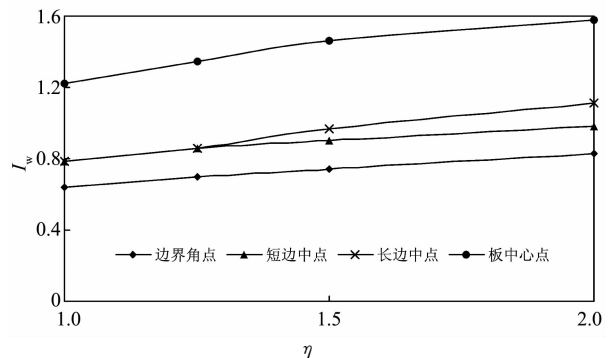


图9 板边界角点、短边中点、长边中点和板中心点的挠度随 \$\eta\$ 的变化曲线

Fig. 9 The settlement-\$\eta\$ curves of the boundary angular dot, the midpoints of the long side and the wide side, and the center of the plate

4 结论

本文基于更加符合工程实际的层状地基模型,采用边界单元法来求解地基与弹性薄板的共同作用问题,并通过与已有文献结果对比,验证了本文理论与计算程序的正确性. 数值分析结果表明:

1) 方形基础薄板情况下,离板中心越近,垂直于 \$y(x)\$ 方向、距离相等的 2 条线段的竖向位移差越小,且其随板-土刚度比减小而减小.

2) 随着长宽比的增大,短边中点与长边中点的挠度差也增大;但板中心点与长边中点位移差变化

不明显,且短边中心与边界角点位移差也有相类似的规律。

相比于有限元、有限层等方法,边界单元法能将求解过程的维数降低一维,并具有计算时间短、精度高等优点。另外,基于层状地基的动力解析层元解^[19]及达朗贝尔原理^[9],还可进一步将本文工作拓展,用以分析层状地基上弹性薄板的动力响应。

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