不同模量理论广义弹性定律的深入研究

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摘　要：传统不同模量理论中基于主应力方向建立的本构方程，仅能表述主应力方向的应力应变关系，并未体现出其他方向的应力应变特性，不能有效表征拉压不同模量问题的力学本质.基于此，在主应力方向的本构方程基础上，利用应力及应变的转轴公式，推导了基于不同直角坐标系下的拉压不同模量本构方程的具体形式，也即广义弹性定律.经理论验证，此广义弹性定律揭示了拉压不同模量问题既是非线性问题也体现出各向异性的力学性质；并且在拉压模量相等时可以回退到经典弹性理论本构方程，而基于主应力方向建立的本构方程是广义弹性定律中的特例. 针对不同模量理论中不甚明晰的剪切模量和泊松比-弹性模量比值的假设，应用所得到的广义弹性定律对纯剪应力状态进行了力学分析，分析表明：在基于最大或最小剪应力方向的直角坐标系下，剪应力与剪应变成线性关系，剪切模量保持不变；并结合微元体纯剪变形的几何关系，证明了假设即拉泊松比与拉模量之比等于压泊松比与压模量之比在纯剪受力状态下是自然满足的.

关键词：弹性理论；不同模量；本构方程；主应力；纯剪

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Theoretical Study on Generalized Elastic Laws of Elastic Theory with Different Modulus

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Abstract：In classical elasticity theory with different modulus, the constitutive equations based on the direction of principal stress can only represent the relationship between the principal stress and principal strain in the main stress direction and cannot reflect the stress−strain behavior in other directions, and the mechanical essence of the problem on different modulus in tension and compression cannot be characterized effectively. Therefore, according to the constitutive equations based on the direction of principal stress, the generalized elastic laws were deduced by the rotation formulas of stress and strain under different Cartesian coordinate system, which are constitutive equations

with different modulus in tension and compression. With theoretical verification, both the nonlinearity and anisotropy property of bi-modulus materials were revealed by the generalized elastic laws. Furthermore, it can also degenerate to the classical bi-modulus elasticity law, which implies that the constitutive law for material with different modulus in tension and compression is special cases of the obtained results. With respect to the indistinct issues about the shear modulus and the assumption of the ratios between Poisson's ratio and Young's modulus, bimodulus material point under pure shear state was investigated. It is shown that, in the rectangular coordinate system based on the maximum or minimum shear stress direction, the relation between shear stress and shear strain is linear. In other words, the shear modulus keeps invariant; besides, the hypothesis is proved that the ratio of tensile Poisson's ratio to tensile modulus is equal to the ratio of compressive Poisson's ratio to compressive modulus under pure shear state, combining with the geometric relationship of pure shear deformation in differential element.

Key words: elastic theory; different modulus; constitutive equations; principal stress; pure shear

大量研究[1-3]表明许多材料的模量呈现拉伸模量与压缩模量不等的性质,导致工程结构的力学响应也具有拉压不同特性.1941 年 Timoshenko[4]首次提出了双模量材料概念.1965 年,C.A.阿姆巴尔楚米扬(Амбарцумян С.А.)[5]再次提出双模量概念,并将其拓展到二维平面及三维空间问题,并于 1982 年编著了第一本关于拉压不同模量问题的专著,建立了基于拉压模量差异的本构理论并称之为不同模量理论. 此后国内外学者都对此问题进行了深入研究,如 Jones[6]将第 2 类区域的柔度矩阵非对角线的柔度系数按主应力绝对值的大小加权计算得到,这样保证了材料的柔度系数矩阵为对称矩阵,其矩阵中的柔度系数不仅与主应力的符号有关,且随着主应力大小的变化而改变,但是此法的理论依据不足;1987 年,Vijayakumar 等[7]从理论上提出把计算模型分成小块子矩阵,将受拉和受压区划分为更细的区域来进行分析,此法基于拉压分块或分区思想,假如构件处于复杂受力状态,拉压分区将会十分困难.

自张允真等[8]将 C.A.阿姆巴尔楚米扬的专著《不同模量理论》于 1986 年翻译出版后,开启了国内学者对此理论的研究热潮,如 Yao 等[9]提出按主应变的正负确定弹性本构矩阵系数的方法;姚文娟等[10-11]基于平截面假设得到了结构中性层判据定理,并推导了不同模量柱、梁、挡土墙等二维受力构件的解析解;何晓婷等[12]放弃了平截面假设,推导了均布荷载简支梁的解析解;吴晓等[13]采用能量法和变分法研究了不同模量板的弯曲问题等. 数值解方面:张允真等[14]在国内首次提出了拉压模量不同的有限元方法,国内学者以此为基础进行了深入研究,如杨海天等[15]提出了有限元计算的初应力迭代

法;刘相斌等[16]提出了加权收敛因子;He 等[17]从理论上推导出了剪切模量通式的具体形式,并提出了具有物理意义的收敛因子;张洪武等[18]通过建立含参变量的拉压不同模量理论的统一本构方程,基于参变量变分原理将拉压模量不同问题转化为互补问题进行求解;Du 等[19-20]通过引入内变量,给出了拉压不同模量材料统一的本构关系和能量表达式,进而发展了系列变分原理和界限理论. 他们还证明了拉压不同模量问题的势能泛函为严格的凸函数,具有解的唯一性和半线性,并基于 Newton-Raphson 算法思想,提出了求解拉压不同模量问题的切线本构算法,此算法对复杂结构及复杂受力也具有高效的收敛效率.

总的来说,国内外绝大部分研究几乎都是将 C.A.阿姆巴尔楚米扬建立的不同模量理论应用到具体问题中去,得到了符合某些特殊受力工况下的解析解或数值解,而对理论的本构模型研究较少且均是在 C.A. 阿姆巴尔楚米扬建立的理论框架下进行的一定改进,同时大部分研究者在应用不同模量理论进行求解时,均沿用了 C.A.阿姆巴尔楚米扬提出的拉压模量与拉压泊松比之间的关系假设即 $\mu^+/E^+ = \mu^-/E^-$. 基于此假设会使材料的柔度矩阵或弹性矩阵变为对称矩阵,使计算推导过程大为简化,尤其对于有限元等数值计算,节约计算成本明显.然而此假设正确与否并未得到理论或试验证明.

基于此,本文在拉压不同模量理论原始定义的本构方程基础上,利用应力及应变在不同直角坐标系下的转轴公式,推导拉压不同模量问题的广义弹性定律,以理清不同模量问题的力学本质,并应用广义弹性定律来证明假设 $\mu^+/E^+ = \mu^-/E^-$的正确性,以期为拉压不同模量问题的求解提供参考或新的思路.

1 不同模量理论广义弹性定律的推导

　　拉压不同模量理论的研究对象是固体和连续体,认为物体是匀质和各向同性的,且基于小变形假设.C.A.阿姆巴尔楚米扬指出:对于不同弹性模量的大部分材料,应力与应变的关系曲线可用两条直线来表示,用这种分段直线函数来表示的简化本构关系,具有足够的精度,完全满足工程应用的要求.在双直线模型中,材料的本构关系分为受拉与受压两种情况:受拉时取拉模量 E^+ 及拉泊松比 μ^+,受压时取压模量 E^- 和压泊松比 μ^-,以此建立了基于主应力方向的本构方程如下:

$$\begin{vmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\gamma \end{vmatrix} = A \begin{vmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\gamma \end{vmatrix}, A = \begin{vmatrix} 1/E^\alpha & -\mu^\beta/E^\beta & -\mu^\gamma/E^\gamma \\ -\mu^\alpha/E^\alpha & 1/E^\beta & -\mu^\gamma/E^\gamma \\ -\mu^\alpha/E^\alpha & -\mu^\beta/E^\beta & 1/E^\gamma \end{vmatrix} \quad (1)$$

式中:$\varepsilon_\alpha, \varepsilon_\beta, \varepsilon_\gamma$ 为主应变;$\sigma_\alpha, \sigma_\beta, \sigma_\gamma$ 为主应力;A 为柔度矩阵;模量 E 和泊松比 μ 由各自对应相乘的主应力正负性质确定,若 $\sigma_\alpha > 0$,则模量 E^α 和泊松比 μ^α 取 E^+ 及 μ^+,反之取 E^- 及 μ^-.由式(1)可知,当 3 个主应力都为正或都为负时,其本构方程与现有经典弹性理论相同,称为不同模量问题的第 1 类区域.当 3 个主应力正负不完全相同时,其本构方程则有明显的差别, 称为不同模量问题的第 2 类区域, 如 $\sigma_\alpha > 0, \sigma_\beta < 0, \sigma_\gamma > 0$,则柔度矩阵 A 为:

$$A = \begin{vmatrix} 1/E^+ & -\mu^-/E^- & -\mu^+/E^+ \\ -\mu^+/E^+ & 1/E^- & -\mu^+/E^+ \\ -\mu^+/E^+ & -\mu^-/E^- & 1/E^+ \end{vmatrix} \quad (2)$$

　　由式(1)和(2)可知,若假设 $\mu^+/E^+ = \mu^-/E^-$,则无论主应力正负如何组合,则 A 都为对称矩阵.由于我们一般是在普通的直角坐标系下进行力学计算,因此需要将建立在主应力方向的本构方程转化为普通直角坐标系下的本构方程,以便于实际应用.对于直角坐标系的 x、y、z 3 轴与主应力($\sigma_\alpha, \sigma_\beta, \sigma_\gamma$)方向之间的方向余弦如表 1 所示,它们之间的关系如式(3)所示.

表 1 主应力与直角坐标轴之间的方向余弦
Tab.1 Direction cosine between the principal stress and rectangular coordinate axis

	α	β	γ
x	l_1	m_1	n_1
y	l_2	m_2	n_2
z	l_3	m_3	n_3

$$\begin{cases} l_1^2 + l_2^2 + l_3^2 = 1 & l_1m_1 + l_2m_2 + l_3m_3 = 0 \\ m_1^2 + m_2^2 + m_3^2 = 1 & l_1n_1 + l_2n_2 + l_3n_3 = 0 \\ n_1^2 + n_2^2 + n_3^2 = 1 & m_1n_1 + m_2n_2 + m_3n_3 = 0 \\ l_1^2 + m_1^2 + n_1^2 = 1 & l_1l_2 + m_1m_2 + n_1n_2 = 0 \\ l_2^2 + m_2^2 + n_2^2 = 1 & l_1l_3 + m_1m_3 + n_1n_3 = 0 \\ l_3^2 + m_3^2 + n_3^2 = 1 & l_2l_3 + m_2m_3 + n_2n_3 = 0 \end{cases} \quad (3)$$

　　依据应力和应变张量的转轴公式及工程剪应变与应变分量的关系有:

$$\begin{cases} \sigma_x = l_1^2\sigma_\alpha + m_1^2\sigma_\beta + n_1^2\sigma_\gamma \\ \sigma_y = l_2^2\sigma_\alpha + m_2^2\sigma_\beta + n_2^2\sigma_\gamma \\ \sigma_z = l_3^2\sigma_\alpha + m_3^2\sigma_\beta + n_3^2\sigma_\gamma \\ \tau_{xy} = l_1l_2\sigma_\alpha + m_1m_2\sigma_\beta + n_1n_2\sigma_\gamma \\ \tau_{yz} = l_2l_3\sigma_\alpha + m_2m_3\sigma_\beta + n_2n_3\sigma_\gamma \\ \tau_{xz} = l_1l_3\sigma_\alpha + m_1m_3\sigma_\beta + n_1n_3\sigma_\gamma \end{cases} \quad (4)$$

$$\begin{cases} \varepsilon_x = l_1^2\varepsilon_\alpha + m_1^2\varepsilon_\beta + n_1^2\varepsilon_\gamma \\ \varepsilon_y = l_2^2\varepsilon_\alpha + m_2^2\varepsilon_\beta + n_2^2\varepsilon_\gamma \\ \varepsilon_z = l_3^2\varepsilon_\alpha + m_3^2\varepsilon_\beta + n_3^2\varepsilon_\gamma \\ \gamma_{xy} = 2(l_1l_2\varepsilon_\alpha + m_1m_2\varepsilon_\beta + n_1n_2\varepsilon_\gamma) \\ \gamma_{yz} = 2(l_2l_3\varepsilon_\alpha + m_2m_3\varepsilon_\beta + n_2n_3\varepsilon_\gamma) \\ \gamma_{xz} = 2(l_1l_3\varepsilon_\alpha + m_1m_3\varepsilon_\beta + n_1n_3\varepsilon_\gamma) \end{cases} \quad (5)$$

$$\begin{cases} \sigma_\alpha = l_1^2\sigma_x + l_2^2\sigma_y + l_3^2\sigma_z + \\ \qquad 2l_1l_2\tau_{xy} + 2l_2l_3\tau_{yz} + 2l_1l_3\tau_{zx} \\ \sigma_\beta = m_1^2\sigma_x + m_2^2\sigma_y + m_3^2\sigma_z + \\ \qquad 2m_1m_2\tau_{xy} + 2m_2m_3\tau_{yz} + 2m_1m_3\tau_{zx} \\ \sigma_\gamma = n_1^2\sigma_x + n_2^2\sigma_y + n_3^2\sigma_z + \\ \qquad 2n_1n_2\tau_{xy} + 2n_2n_3\tau_{yz} + 2n_1n_3\tau_{zx} \end{cases} \quad (6)$$

　　将式(4)及式(5)代入式(1),化简可得如下本构方程为:

$$\begin{cases} \varepsilon_x = \dfrac{l_1^2\sigma_\alpha}{2G_\alpha} + \dfrac{m_1^2\sigma_\beta}{2G_\beta} + \dfrac{n_1^2\sigma_\gamma}{2G_\gamma} + A_\alpha\sigma_\alpha + A_\beta\sigma_\beta + A_\gamma\sigma_\gamma \\[2mm] \varepsilon_y = \dfrac{l_2^2\sigma_\alpha}{2G_\alpha} + \dfrac{m_2^2\sigma_\beta}{2G_\beta} + \dfrac{n_2^2\sigma_\gamma}{2G_\gamma} + A_\alpha\sigma_\alpha + A_\beta\sigma_\beta + A_\gamma\sigma_\gamma \\[2mm] \varepsilon_z = \dfrac{l_3^2\sigma_\alpha}{2G_\alpha} + \dfrac{m_3^2\sigma_\beta}{2G_\beta} + \dfrac{n_3^2\sigma_\gamma}{2G_\gamma} + A_\alpha\sigma_\alpha + A_\beta\sigma_\beta + A_\gamma\sigma_\gamma \\[2mm] \gamma_{xy} = \dfrac{l_1l_2\sigma_\alpha}{G_\alpha} + \dfrac{m_1m_2\sigma_\beta}{G_\beta} + \dfrac{n_1n_2\sigma_\gamma}{G_\gamma} \\[2mm] \gamma_{yz} = \dfrac{l_2l_3\sigma_\alpha}{G_\alpha} + \dfrac{m_2m_3\sigma_\beta}{G_\beta} + \dfrac{n_2n_3\sigma_\gamma}{G_\gamma} \\[2mm] \gamma_{xz} = \dfrac{l_1l_3\sigma_\alpha}{G_\alpha} + \dfrac{m_1m_3\sigma_\beta}{G_\beta} + \dfrac{n_1n_3\sigma_\gamma}{G_\gamma} \end{cases} \quad (7)$$

式中：

$$\begin{cases} G_\alpha = E^\alpha/[2(1+\mu^\alpha)], A_\alpha = -\mu^\alpha/E^\alpha \\ G_\beta = E^\beta/[2(1+\mu^\beta)], A_\beta = -\mu^\beta/E^\beta \\ G_\gamma = E^\gamma/[2(1+\mu^\gamma)], A_\gamma = -\mu^\gamma/E^\gamma \end{cases} \tag{8}$$

当拉压模量相等时，有

$$\begin{cases} G_\alpha = G_\beta = G_\gamma = E/[2(1+\mu)] \\ A_\alpha = A_\beta = A_\gamma = -\mu/E \end{cases} \tag{9}$$

式(7)即退化为经典的胡克定律. 当拉压模量不等时，对于第 1 类区域，式(5)的形式与经典的胡克定律类似，只不过当 $\sigma_\alpha > 0, \sigma_\beta > 0, \sigma_\gamma > 0$ 时，有

$$\begin{cases} G_\alpha = G_\beta = G_\gamma = E^+/[2(1+\mu^+)] = G^+ \\ A_\alpha = A_\beta = A_\gamma = -\mu^+/E^+ = A^+ \end{cases} \tag{10}$$

当 $\sigma_\alpha < 0, \sigma_\beta < 0, \sigma_\gamma < 0$ 时，有

$$\begin{cases} G_\alpha = G_\beta = G_\gamma = E^-/[2(1+\mu^-)] = G^- \\ A_\alpha = A_\beta = A_\gamma = -\mu^-/E^- = A^- \end{cases} \tag{11}$$

对于第 2 类区域，以 $\sigma_\alpha < 0, \sigma_\beta < 0, \sigma_\gamma < 0$ 为例，弹性本构方程可整理为如下形式：

$$\begin{cases} \varepsilon_x = \dfrac{\sigma_x}{2G^+} + A^+\Theta + \left(\dfrac{1}{G^-} - \dfrac{1}{G^+}\right)\dfrac{m_1^2\sigma_\beta}{2} + (A^- - A^+)\sigma_\beta \\[2mm] \varepsilon_y = \dfrac{\sigma_y}{2G^+} + A^+\Theta + \left(\dfrac{1}{G^-} - \dfrac{1}{G^+}\right)\dfrac{m_2^2\sigma_\beta}{2} + (A^- - A^+)\sigma_\beta \\[2mm] \varepsilon_z = \dfrac{\sigma_z}{2G^+} + A^+\Theta + \left(\dfrac{1}{G^-} - \dfrac{1}{G^+}\right)\dfrac{m_3^2\sigma_\beta}{2} + (A^- - A^+)\sigma_\beta \\[2mm] \gamma_{xy} = \dfrac{\tau_{xy}}{G^+} + \left(\dfrac{1}{G^-} - \dfrac{1}{G^+}\right)m_1 m_2\sigma_\beta \\[2mm] \gamma_{yz} = \dfrac{\tau_{yz}}{G^+} + \left(\dfrac{1}{G^-} - \dfrac{1}{G^+}\right)m_2 m_3\sigma_\beta \\[2mm] \gamma_{xz} = \dfrac{\tau_{xz}}{G^+} + \left(\dfrac{1}{G^-} - \dfrac{1}{G^+}\right)m_1 m_3\sigma_\beta \end{cases} \tag{12}$$

式(12)中 Θ 为体应力. 由式(7)和式(12)可知，基于普通直角坐标系下拉压模量不等的本构方程，与经典的等模量本构方程已完全不同，应力与应变不再是简单的线性关系. 由式(12)可知，除了熟知的经典弹性关系中的线性项外，还有非线性项；且其线性项的系数不再是常量，而是随着主应力正负的变化而变化. 非线性项则与主应力的正负、大小及方向有关.

基于以上分析，将式(6)代入式(7)，将其整理为任意方向直角坐标系下的本构方程为：

$$\begin{aligned} \varepsilon_x = &\left[\left(\frac{l_1^2}{2G_\alpha}+A_\alpha\right)l_1^2 + \left(\frac{m_1^2}{2G_\beta}+A_\beta\right)m_1^2 + \left(\frac{n_1^2}{2G_\gamma}+A_\gamma\right)n_1^2\right]\sigma_x + 2\left[\left(\frac{l_1^2}{2G_\alpha}+A_\alpha\right)l_1 l_2 + \left(\frac{m_1^2}{2G_\beta}+A_\beta\right)m_1 m_2 + \left(\frac{n_1^2}{2G_\gamma}+A_\gamma\right)n_1 n_2\right]\tau_{xy} + \\ &\left[\left(\frac{l_1^2}{2G_\alpha}+A_\alpha\right)l_2^2 + \left(\frac{m_1^2}{2G_\beta}+A_\beta\right)m_2^2 + \left(\frac{n_1^2}{2G_\gamma}+A_\gamma\right)n_2^2\right]\sigma_y + 2\left[\left(\frac{l_1^2}{2G_\alpha}+A_\alpha\right)l_2 l_3 + \left(\frac{m_1^2}{2G_\beta}+A_\beta\right)m_2 m_3 + \left(\frac{n_1^2}{2G_\gamma}+A_\gamma\right)n_2 n_3\right]\tau_{yz} + \\ &\left[\left(\frac{l_1^2}{2G_\alpha}+A_\alpha\right)l_3^2 + \left(\frac{m_1^2}{2G_\beta}+A_\beta\right)m_3^2 + \left(\frac{n_1^2}{2G_\gamma}+A_\gamma\right)n_3^2\right]\sigma_z + 2\left[\left(\frac{l_1^2}{2G_\alpha}+A_\alpha\right)l_1 l_3 + \left(\frac{m_1^2}{2G_\beta}+A_\beta\right)m_1 m_3 + \left(\frac{n_1^2}{2G_\gamma}+A_\gamma\right)n_1 n_3\right]\tau_{xz} \end{aligned} \tag{13}$$

$$\begin{aligned} \varepsilon_y = &\left[\left(\frac{l_2^2}{2G_\alpha}+A_\alpha\right)l_1^2 + \left(\frac{m_2^2}{2G_\beta}+A_\beta\right)m_1^2 + \left(\frac{n_2^2}{2G_\gamma}+A_\gamma\right)n_1^2\right]\sigma_x + 2\left[\left(\frac{l_2^2}{2G_\alpha}+A_\alpha\right)l_1 l_2 + \left(\frac{m_2^2}{2G_\beta}+A_\beta\right)m_1 m_2 + \left(\frac{n_2^2}{2G_\gamma}+A_\gamma\right)n_1 n_2\right]\tau_{xy} + \\ &\left[\left(\frac{l_2^2}{2G_\alpha}+A_\alpha\right)l_2^2 + \left(\frac{m_2^2}{2G_\beta}+A_\beta\right)m_2^2 + \left(\frac{n_2^2}{2G_\gamma}+A_\gamma\right)n_2^2\right]\sigma_y + 2\left[\left(\frac{l_2^2}{2G_\alpha}+A_\alpha\right)l_2 l_3 + \left(\frac{m_2^2}{2G_\beta}+A_\beta\right)m_2 m_3 + \left(\frac{n_2^2}{2G_\gamma}+A_\gamma\right)n_2 n_3\right]\tau_{yz} + \\ &\left[\left(\frac{l_2^2}{2G_\alpha}+A_\alpha\right)l_3^2 + \left(\frac{m_2^2}{2G_\beta}+A_\beta\right)m_3^2 + \left(\frac{n_2^2}{2G_\gamma}+A_\gamma\right)n_3^2\right]\sigma_z + 2\left[\left(\frac{l_2^2}{2G_\alpha}+A_\alpha\right)l_1 l_3 + \left(\frac{m_2^2}{2G_\beta}+A_\beta\right)m_1 m_3 + \left(\frac{n_2^2}{2G_\gamma}+A_\gamma\right)n_1 n_3\right]\tau_{xz} \end{aligned} \tag{14}$$

$$\begin{aligned} \varepsilon_z = &\left[\left(\frac{l_3^2}{2G_\alpha}+A_\alpha\right)l_1^2 + \left(\frac{m_3^2}{2G_\beta}+A_\beta\right)m_1^2 + \left(\frac{n_3^2}{2G_\gamma}+A_\gamma\right)n_1^2\right]\sigma_x + 2\left[\left(\frac{l_3^2}{2G_\alpha}+A_\alpha\right)l_1 l_2 + \left(\frac{m_3^2}{2G_\beta}+A_\beta\right)m_1 m_2 + \left(\frac{n_3^2}{2G_\gamma}+A_\gamma\right)n_1 n_2\right]\tau_{xy} + \\ &\left[\left(\frac{l_3^2}{2G_\alpha}+A_\alpha\right)l_2^2 + \left(\frac{m_3^2}{2G_\beta}+A_\beta\right)m_2^2 + \left(\frac{n_3^2}{2G_\gamma}+A_\gamma\right)n_2^2\right]\sigma_y + 2\left[\left(\frac{l_3^2}{2G_\alpha}+A_\alpha\right)l_2 l_3 + \left(\frac{m_3^2}{2G_\beta}+A_\beta\right)m_2 m_3 + \left(\frac{n_3^2}{2G_\gamma}+A_\gamma\right)n_2 n_3\right]\tau_{yz} + \\ &\left[\left(\frac{l_3^2}{2G_\alpha}+A_\alpha\right)l_3^2 + \left(\frac{m_3^2}{2G_\beta}+A_\beta\right)m_3^2 + \left(\frac{n_3^2}{2G_\gamma}+A_\gamma\right)n_3^2\right]\sigma_z + 2\left[\left(\frac{l_3^2}{2G_\alpha}+A_\alpha\right)l_1 l_3 + \left(\frac{m_3^2}{2G_\beta}+A_\beta\right)m_1 m_3 + \left(\frac{n_3^2}{2G_\gamma}+A_\gamma\right)n_1 n_3\right]\tau_{xz} \end{aligned} \tag{15}$$

$$\gamma_{xy} = \left[\frac{l_1 l_2}{G_\alpha} l_1^2 + \frac{m_1 m_2}{G_\beta} m_1^2 + \frac{n_1 n_2}{G_\gamma} n_1^2 \right] \sigma_x + \left[\frac{l_1 l_2}{G_\alpha} l_2^2 + \frac{m_1 m_2}{G_\beta} m_2^2 + \frac{n_1 n_2}{G_\gamma} n_2^2 \right] \sigma_y +$$

$$\left[\frac{l_1 l_2}{G_\alpha} l_3^2 + \frac{m_1 m_2}{G_\beta} m_3^2 + \frac{n_1 n_2}{G_\gamma} n_3^2 \right] \sigma_z + \left[\frac{l_1 l_2}{G_\alpha} 2l_1 l_2 + \frac{m_1 m_2}{G_\beta} 2m_1 m_2 + \frac{n_1 n_2}{G_\gamma} 2n_1 n_2 \right] \tau_{xy} +$$

$$\left[\frac{l_1 l_2}{G_\alpha} 2l_1 l_3 + \frac{m_1 m_2}{G_\beta} 2m_1 m_3 + \frac{n_1 n_2}{G_\gamma} 2n_1 n_3 \right] \tau_{xz} + \left[\frac{l_1 l_2}{G_\alpha} 2l_2 l_3 + \frac{m_1 m_2}{G_\beta} 2m_2 m_3 + \frac{n_1 n_2}{G_\gamma} 2n_2 n_3 \right] \tau_{yz} \tag{16}$$

$$\gamma_{xy} = \left[\frac{l_1 l_3}{G_\alpha} l_1^2 + \frac{m_1 m_3}{G_\beta} m_1^2 + \frac{n_1 n_3}{G_\gamma} n_1^2 \right] \sigma_x + \left[\frac{l_1 l_3}{G_\alpha} l_2^2 + \frac{m_1 m_3}{G_\beta} m_2^2 + \frac{n_1 n_3}{G_\gamma} n_2^2 \right] \sigma_y +$$

$$\left[\frac{l_1 l_3}{G_\alpha} l_3^2 + \frac{m_1 m_3}{G_\beta} m_3^2 + \frac{n_1 n_3}{G_\gamma} n_3^2 \right] \sigma_z + \left[\frac{l_1 l_3}{G_\alpha} 2l_1 l_2 + \frac{m_1 m_3}{G_\beta} 2m_1 m_2 + \frac{n_1 n_3}{G_\gamma} 2n_1 n_2 \right] \tau_{xy} +$$

$$\left[\frac{l_1 l_3}{G_\alpha} 2l_1 l_3 + \frac{m_1 m_3}{G_\beta} 2m_1 m_3 + \frac{n_1 n_3}{G_\gamma} 2n_1 n_3 \right] \tau_{xz} + \left[\frac{l_1 l_3}{G_\alpha} 2l_2 l_3 + \frac{m_1 m_3}{G_\beta} 2m_2 m_3 + \frac{n_1 n_3}{G_\gamma} 2n_2 n_3 \right] \tau_{yz} \tag{17}$$

$$\gamma_{xy} = \left[\frac{l_2 l_3}{G_\alpha} l_1^2 + \frac{m_2 m_3}{G_\beta} m_1^2 + \frac{n_2 n_3}{G_\gamma} n_1^2 \right] \sigma_x + \left[\frac{l_2 l_3}{G_\alpha} l_2^2 + \frac{m_2 m_3}{G_\beta} m_2^2 + \frac{n_2 n_3}{G_\gamma} n_2^2 \right] \sigma_y +$$

$$\left[\frac{l_2 l_3}{G_\alpha} l_3^2 + \frac{m_2 m_3}{G_\beta} m_3^2 + \frac{n_2 n_3}{G_\gamma} n_3^2 \right] \sigma_z + \left[\frac{l_2 l_3}{G_\alpha} 2l_1 l_2 + \frac{m_2 m_3}{G_\beta} 2m_1 m_2 + \frac{n_2 n_3}{G_\gamma} 2n_1 n_2 \right] \tau_{xy} +$$

$$\left[\frac{l_2 l_3}{G_\alpha} 2l_1 l_3 + \frac{m_2 m_3}{G_\beta} 2m_1 m_3 + \frac{n_2 n_3}{G_\gamma} 2n_1 n_3 \right] \tau_{xz} + \left[\frac{l_2 l_3}{G_\alpha} 2l_2 l_3 + \frac{m_2 m_3}{G_\beta} 2m_2 m_3 + \frac{n_2 n_3}{G_\gamma} 2n_2 n_3 \right] \tau_{yz} \tag{18}$$

由式(13)~式(18)可知,此本构方程中所有系数项均不含主应力或主应变,因此主要表征普通直角坐标系方向下的应力应变关系,即广义弹性定律.且此本构方程满足回退特性,当拉压模量相等时,有

$$\varepsilon_x = \left(\frac{l_1^2}{2G} + A \right)(l_1^2 \sigma_x + l_2^2 \sigma_y + l_3^2 \sigma_z + 2l_1 l_2 \tau_{xy} + 2l_1 l_3 \tau_{xz} + 2l_2 l_3 \tau_{yz}) + \left(\frac{m_1^2}{2G} + A \right)(m_1^2 \sigma_x + m_2^2 \sigma_y + m_3^2 \sigma_z +$$

$$2m_1 m_2 \tau_{xy} + 2m_1 m_3 \tau_{xz} + 2m_2 m_3 \tau_{yz}) + \left(\frac{n_1^2}{2G} + A \right)(n_1^2 \sigma_x + n_2^2 \sigma_y + n_3^2 \sigma_z + 2n_1 n_2 \tau_{xy} + 2n_1 n_3 \tau_{xz} + 2n_2 n_3 \tau_{yz}) =$$

$$\left(\frac{l_1^2}{2G} + A \right)\sigma_\alpha + \left(\frac{m_1^2}{2G} + A \right)\sigma_\beta + \left(\frac{n_1^2}{2G} + A \right)\sigma_\gamma = \frac{1}{2G}(l_1^2 \sigma_\alpha + m_1^2 \sigma_\beta + n_1^2 \sigma_\gamma) + A(\sigma_\alpha + \sigma_\beta + \sigma_\gamma) =$$

$$\frac{\sigma_x}{2G} + A(\sigma_x + \sigma_y + \sigma_z) = \frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E} - \frac{\mu\sigma_z}{E} \tag{19}$$

其他 5 个方程同理可证明.基于此,具有普遍意义的本构方程可写为如下形式:

$$\begin{cases} \sigma = (\sigma_x \ \ \sigma_y \ \ \sigma_z \ \ \tau_{xy} \ \ \tau_{yz} \ \ \tau_{xz})^T \\ \varepsilon = (\varepsilon_x \ \ \varepsilon_y \ \ \varepsilon_z \ \ \gamma_{xy} \ \ \gamma_{yz} \ \ \gamma_{xz})^T \end{cases} \tag{20}$$

$$\varepsilon = C\sigma, \ C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \tag{21}$$

矩阵 C 可称为广义柔度矩阵,矩阵中元素分别对应式(13)~式(18)中与每个对应应力相乘的多项式,当直角坐标轴与主应力方向重合时,即:$l_1 = m_2 = n_3 = 1, l_2 = l_3 = m_1 = m_3 = n_1 = n_2 = 0$ 时,则有广义柔度矩阵各元素为:$c_{11} = 1/E_\alpha, c_{22} = 1/E_\beta, c_{33} = 1/E_\gamma, c_{21} = c_{31} = -\mu_\alpha/E_\alpha, c_{12} = c_{32} = -\mu^\beta/E^\beta, c_{13} = c_{23} = -\mu^\gamma/E^\gamma$,矩阵 C 中的第 4、5、6 行及第 4、5、6 列元素均等于 0, 形式如式(22)所示.

$$C = \begin{bmatrix} 1/E^\alpha & -\mu^\beta/E^\beta & -\mu^\gamma/E^\gamma & 0 & 0 & 0 \\ -\mu^\alpha/E^\alpha & 1/E^\beta & -\mu^\gamma/E^\gamma & 0 & 0 & 0 \\ -\mu^\alpha/E^\alpha & -\mu^\beta/E^\beta & 1/E^\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

由式(22)可知,广义柔度矩阵回退到了不同模量理论的原始定义柔度矩阵(见式(1)),这表明本文建立的广义弹性定律符合拉压模量的原始定义,且更具普遍性;基于主应力方向建立的本构方程,是广义弹性定律中的特例,由于剪应力及剪应变为0,避免了对剪应力及剪应变的讨论,其柔度矩阵变为 3×3 阶矩阵,使得本构方程得到简化.

文献[17]指出:由基于主方向上的本构方程直接推导复杂应力状况下的本构矩阵不满足回退特性,必须要加上剪切通项即当直角坐标系建立在主应力方向时,柔度系数 c_{44}、c_{55}、c_{66} 必须要赋予合适的非零值,才能满足拉压回退特性.本文推导表明,在利用应力应变的转轴公式后,广义弹性本构方程的拉压回退特性是满足的,与是否补全剪切通项无关.

2 拉压不同模量问题力学性质讨论

由具有拉压不同模量特性的材料组成的结构在复杂受力状态下,任意一点的受力状态可分为如下 3 种情形:1)该点的 3 个主应力同为正或同为负,即所谓的第 1 类区域,该区域内任一点的本构方程与各向同性本构方程类似;2)3 个主应力符号不完全相同,但主应力方向刚好与整体的直角坐标轴方向相同,此时该点的本构方程可简化为原始定义的本构方程即式(1),这与正交各向异性的本构方程类似,其主轴与该点的主应力方向相同;3)3 个主应力符号不完全相同,主应力方向与整体的直角坐标轴也不重合,复杂受力状态下此类区域一般占绝大比例,其本构方程形式即为式(13)~(18),此类区域中各点的主应力大小及方向一般都各不相同,即任一点的 3 个主应力及其方向余弦均不相等.则由式(13)~(18)可知,该区域各点的广义本构方程形式上一样,但对应的各系数项均不相等.且从本构方程的形式上看,正应变不只与正应力有关,也受剪应力的影响;剪应变也不只与对应的剪应力有关,也受 3 个正应力和其他两个方向剪应力的影响.因此,从广义弹性本构方程角度来讲,该区域的本构方程与各向异性的本构方程类似.

由以上两节分析可得,即使在各个应力状态的主应力空间内,本构关系具有线弹性形式,但对于整个拉压不同模量弹性系统,其本构方程由于决定于主应力正负而表现出非线性性质;并且在一般直角坐标轴中,本构关系依赖于该方向与主应力方向的夹角,形式上具有各向异性特征,显然没有固定的材料主轴.因此拉压不同结构的力学行为受其应力状态及整体坐标轴方向的影响,已属非线性及各向异性范畴.

3 纯剪应力状态受力分析

应用广义弹性定律,对二维纯剪应力状态进行分析,以正方形微元体为例,其受力模式如图 1 所示.
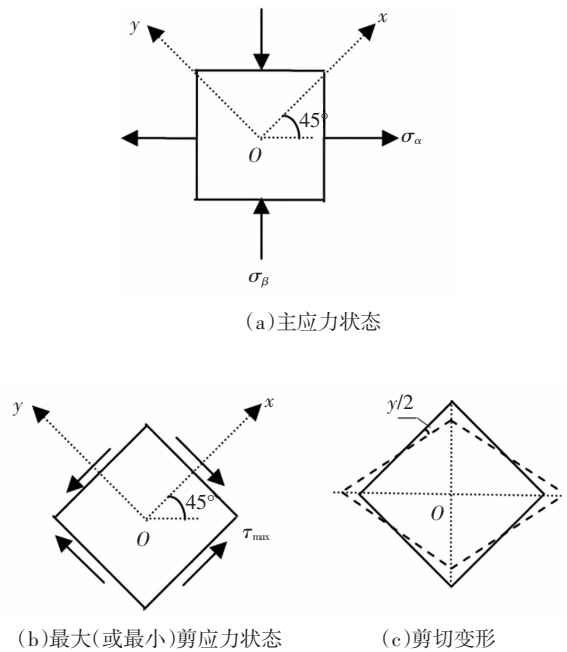

(a)主应力状态

(b)最大(或最小)剪应力状态　　(c)剪切变形

图 1　纯剪应力状态示意图

Fig.1　Schematic diagram on the stress state of pure shear

由图 1(a)及图 1(b)可知,直角坐标轴与主应力的最小夹角为 45°,则主应力方向与直角坐标之间的方向余弦分别为:$l_1 = \sqrt{2}/2, l_2 = \sqrt{2}/2, m_1 = -\sqrt{2}/2, m_2 = \sqrt{2}/2$,设 $\sigma_\alpha > 0, \sigma_\beta > 0$,则

$$\begin{cases} G_\alpha = G^+, G_\beta = G^- \\ A_\alpha = A^+, A_\beta = A^- \end{cases} \tag{23}$$

由式(12)可得二维状态下的本构方程为:

$$\begin{cases} \varepsilon_x = \dfrac{\sigma_x}{2G^+} + A^+\Theta + \left(\dfrac{1}{G^-} - \dfrac{1}{G^+}\right)\dfrac{m_1^2\sigma_\beta}{2} + (A^- - A^+)\sigma_\beta \\ \varepsilon_y = \dfrac{\sigma_y}{2G^+} + A^+\Theta + \left(\dfrac{1}{G^-} - \dfrac{1}{G^+}\right)\dfrac{m_2^2\sigma_\beta}{2} + (A^- - A^+)\sigma_\beta \\ \gamma_{xy} = \dfrac{\tau_{xy}}{G^+} + \left(\dfrac{1}{G^-} - \dfrac{1}{G^+}\right)m_1 m_2 \sigma_\beta \end{cases} \tag{24}$$

若 $\sigma_\alpha = \sigma_0, \sigma_\beta = -\sigma_0$,由式(4)和式(24)可得:

$$\left\{\begin{array}{l}\tau_{xy} = \dfrac{1}{2}(\sigma_\beta - \sigma_\alpha) = -\sigma_0 \\[2mm] \gamma_{yx} = \varepsilon_\beta - \varepsilon_\alpha = \dfrac{\sigma_\beta - \sigma_\alpha}{2G^+} + \left(\dfrac{1}{G^-} - \dfrac{1}{G^+}\right)\dfrac{\sigma_\beta}{2}\end{array}\right. \quad (25)$$

整理式（25）可得：

$$\gamma_{xy} = \frac{-2\sigma_0}{2G^+} + \left(\frac{1}{G^-} - \frac{1}{G^+}\right)\frac{-\sigma_0}{2} = \tau_{xy}/\left(\frac{2G^+G^-}{G^++G^-}\right) \quad (26)$$

假如 $\sigma_\alpha = -\sigma_0, \sigma_\beta = \sigma_0$，同理可得：

$$\gamma_{xy} = \frac{2\sigma_0}{2G^-} + \left(\frac{1}{G^+} - \frac{1}{G^-}\right)\frac{\sigma_0}{2} = \tau_{xy}/\left(\frac{2G^+G^-}{G^++G^-}\right) \quad (27)$$

对比式（26）和式（27）可得，纯剪状态下，在基于最大（或最小）剪应力方向的直角坐标系下，剪应力与剪应变成线性关系，剪切模量不随剪应力的大小和正负发生改变.同时表明图1(c)所示微元体中左右两个直角的减小量与上下两个直角的增加量相等.

4　关于 $\mu^+/E^+ = \mu^-/E^-$ 的证明

假设微元体为纯剪应力状态，且 $\sigma_\alpha = -\sigma_0, \sigma_\beta = \sigma_0$，则

$$\sigma_x = \sigma_y = \Theta = 0 \quad (28)$$

由式（24）可得：

$$\varepsilon_x = \varepsilon_y = \varepsilon = \left(\frac{1}{G^-} - \frac{1}{G^+}\right)\frac{m_1^2\sigma_\beta}{2} + (A^- - A^+)\sigma_\beta =$$
$$\frac{\sigma_0}{2}\left(\frac{1+3\mu^+}{E^+} - \frac{1+3\mu^-}{E^-}\right) \quad (29)$$

这表明微元体变形后各边的边长依然相等，结合第3节分析可知，微元体变形后仍然是菱形，且变形后菱形的中心仍然与原中心 O 重合.设微元体变形前边长为 a，对角线长为 d，变形后边长为 a'，则

$$a' = a(1+\varepsilon) = \frac{\sqrt{2}}{2}d(1+\varepsilon) \quad (30)$$

设微元体的角应变为 γ，则由几何关系可知：

$$\left\{\begin{array}{l}\varepsilon_\alpha \dfrac{d}{2} = a'\cos\left(\dfrac{\pi}{4} - \dfrac{\gamma}{2}\right) - \dfrac{d}{2} \\[3mm] \varepsilon_\beta \dfrac{d}{2} = a'\sin\left(\dfrac{\pi}{4} - \dfrac{\gamma}{2}\right) - \dfrac{d}{2}\end{array}\right. \quad (31)$$

将式（30）代入式（31）整理可得：

$$\left\{\begin{array}{l}\varepsilon_\alpha = \sqrt{2}\,(1+\varepsilon)\left(\dfrac{\sqrt{2}}{2}\cos\dfrac{\gamma}{2} + \dfrac{\sqrt{2}}{2}\sin\dfrac{\gamma}{2}\right) - 1 \\[3mm] \varepsilon_\beta = \sqrt{2}\,(1+\varepsilon)\left(\dfrac{\sqrt{2}}{2}\cos\dfrac{\gamma}{2} - \dfrac{\sqrt{2}}{2}\sin\dfrac{\gamma}{2}\right) - 1\end{array}\right.$$
$$(32)$$

由小变形假设，解式（32）可得：

$$\varepsilon = (\varepsilon_\alpha + \varepsilon_\alpha)/2 \quad (33)$$

又由式（1）可得二维受力状态下，基于主应力方向的本构方程为：

$$\left\{\begin{array}{l}\varepsilon_\alpha = \dfrac{\sigma_0}{E^+} + \dfrac{\mu^-\sigma_0}{E^-} \\[3mm] \varepsilon_\beta = -\dfrac{\sigma_0}{E^-} - \dfrac{\mu^+\sigma_0}{E^+}\end{array}\right. \quad (34)$$

将式（34）代入式（33）得：

$$\varepsilon = \frac{\sigma_0}{2}\left(\frac{1-\mu^+}{E^+} - \frac{1-\mu^-}{E^+}\right) \quad (35)$$

结合式（29）和式（35）可得等式：

$$\frac{\sigma_0}{2}\left(\frac{1+3\mu^+}{E^+} - \frac{1+3\mu^-}{E^-}\right) = \frac{\sigma_0}{2}\left(\frac{1-\mu^+}{E^+} - \frac{1-\mu^-}{E^-}\right) \quad (36)$$

化简得：

$$\mu^+/E^+ = \mu^-/E^- \quad (37)$$

由于本文建立的本构方程及推导时并没有采用假设 $\mu^+/E^+ = \mu^-/E^-$，这表明只要材料具有拉压模量不同的性质，且基于拉压模量双线性模型建立的本构方程，在纯剪受力状态下，自然满足 $\mu^+/E^+ = \mu^-/E^-$.同时基于此结论，可大为简化广义柔度矩阵 $\boldsymbol{C}$，并可证明 $\boldsymbol{C}$ 为对称矩阵，这与材料的匀质性假设相符合.

5　结　论

在基于主应力方向的本构方程基础上，利用应力及应变的转轴公式，推导了不同模量理论的广义弹性定律，此定律满足拉压模量相同时的回退特性，且深刻揭示了拉压不同模量问题的各向异性和非线性特性. 应用广义弹性定律对纯剪应力状态进行了力学分析，结果表明：在基于最大（或最小）剪应力方向的直角坐标系下，剪应力与剪应变成线性关系，剪切模量保持不变；并结合微元体纯剪变形的几何关系证明了假设 $\mu^+/E^+ = \mu^-/E^-$ 在纯剪受力状态下是自然满足的.

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